114-E FINAL EXAMINATION

NAME: ___________________________________

Prof. Ghrist : Spring 2017

INSTRUCTIONS:

No book or calculators or notes.

Use a writing utensil and logic.

Show all of your work.
Explain yourself clearly to receive partial credit.

Cheating, or the appearance of cheating, will be dealt with severely.

Good luck!

You may find the following helpful:

Spherical coordinates:

$$x = \rho \cos\theta \ \sin\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

DO NOT

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PROBLEM 1: Fill in the following [short answers]

(A) What are the Lagrange Equations for extremizing a function $F(x)$ subject to the constraint $G(x) = 0$?

(B) What is the interpretation of the curl of a vector field $\vec{F}$ at a point?

(C) Green's Theorem says that for a 1-form on a bounded domain $D \subset \mathbb{R}^2$

$$\int_{\partial D} P\, dx + Q\, dy \quad =$$

(D) The condition on the Hessian (or 2nd derivative) $[Hf]$ for having a local minimum of $f: \mathbb{R}^2 \to \mathbb{R}$ is that…

(E) Green's, Gauss', & Stokes' theorems all have the same appearance when written in the language of differential forms: what is that equation?

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PROBLEM 2: Please compute the following [short answers, no detailed justification needed]

(A) $(3\,dz + 2y\,dx) \wedge (x\,dy - y\,dz) \wedge dz$

ANSWER

(B) $(\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}) \times (\hat{\jmath} - \hat{k})$

ANSWER

(C) the determinant

$$\begin{vmatrix} 1 & -4 & 0 \\ 2 & 3 & 0 \\ 8 & 9 & -2 \end{vmatrix}$$

ANSWER

(D) the derivative of the 1-form

$$\alpha = x^2 dx - xy\,dy$$

ANSWER

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PROBLEM 3: Answer the following:

(A) Write down the chain rule for $f\colon \mathbb{R}^m \to \mathbb{R}^p$ and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ at an input $\underline{a} \in \mathbb{R}^n$. (be careful of the evaluation points!)

(B) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be invertible – that is, there is an inverse $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ so that $f\left(g\left(\underline{x}\right)\right) = \underline{x} = g(f(\underline{x}))$ for all $\underline{x} \in \mathbb{R}^2$ (that is, $f^\circ g$ and $g^\circ f$ is the identity). If I tell you that at an input $\underline{a} \in \mathbb{R}^2$ the derivative of f is

$$[Df]_{\underline{a}} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$$

Then what is the derivative of g at $f(\underline{a})$?

ANSWER

(C) What if I tell you that f is in fact *not* invertible at this input $\underline{a}$? Would you believe me? Why or why not?

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PROBLEM 4: What is the center of mass of the solid described by $z \geq x^2 + y^2$ & $z \leq 1$ and having density $= z$? Be sure to give reasons and/or show work.

ANSWER

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ANSWER

PROBLEM 5: Compute the flux of the vector field

$\vec{F} = (2x - \ sin(2^{yz}))\hat{\imath} + \ (sec(x - 2z))^2\hat{\jmath} + (7\,tan(e^{-xy}))\hat{k}$

through the boundary of the unit ball in $\mathbb{R}^3$ (oriented outwards).

(In case you are panicking about the derivative of secant,
*it is secant*tangent. But you shouldn't be panicking about that...)*

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PROBLEM 6: Let's say you have a factory full of robots that make pizzas (great startup idea! ***pizzabots!***). Assume you have $x > 0$ robots, and each robot can be built to produce $y > 0$ pizzas per hour.

A) What is the total number of pizzas $P(x, y)$ produced per hour? Don't get stuck here; this is easy.

B) Assume the cost of such a factory is $C(x, y) = 3x^2 + 6y$ (it's more expensive to have lots of robots, since the factory has to be bigger, you need more supplies, etc.). For a fixed production of 1000 pizzas per hour, how should you allocate the number, x, and productivity, y, of your pizzabots to minimize cost?

Please use the method of Lagrange to solve and show your work.

ANSWER

DO NOT

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PROBLEM 7: Please compute the following [show some work…]

(A) The divergence of:
$\vec{F} = -y\,\hat{\imath} + yz\,\hat{\jmath} + 6xy\,\hat{k}$

ANSWER

(B) The flux 2-form of the vector field:

ANSWER

$\vec{F} = z\,\hat{\imath} + y^2\,\hat{\jmath} + x\,\hat{k}$

(C) The work $\int \vec{F} \cdot d\underline{x}$ done by the gradient field

ANSWER

$$\vec{F} = \nabla(x + 2y + z^2)$$

along the helical path $x = \cos t$; $y = \sin t$; $z = t$; $t = 0 \ldots 2\pi$

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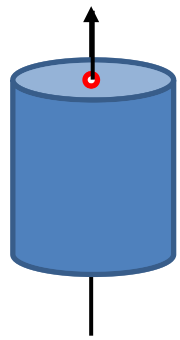
PROBLEM 8: Consider a unit-density solid cylinder of radius R and height h rotated through a central axis as shown.



A) What is the mass M? [no work needed]

ANSWER

B) Using a coordinate frame centered "in the middle" of this body, with the z-axis being the axis of rotation, what is the moment-of-inertia element?

ANSWER

C) Compute the moment of inertia in terms of mass M, and the parameters R and h. Show work!

ANSWER

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PROBLEM 9: Compute the seemingly-impossible integral:

$$\int_0^2 \int_{\frac{x}{2}}^{\frac{x}{2}+1} x^5(2y-x)e^{(2y-x)^2}\,dy\,dx$$

by using the change of coordinates $u = x$, $v = 2y - x$

ANSWER

DO NOT

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PROBLEM 10: Consider the following system of linear equations for x, y, and z:

$$
\begin{aligned}
- y + 3z &= 0 \\
x + 3y - z &= 13 \\
2x + 4y + 3z &= 24
\end{aligned}
$$

A) Write this in matrix form.

B) Solve the equations using row reduction: show all steps!

ANSWER

x =

y =

z =

DO NOT

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PROBLEM 11: Compute the following briefly, showing some work:

A) The equation of a plane through the point $x = 1, y = 2, z = 3$ and perpendicular to the vector $4\,\hat{\imath} + 5\,\hat{\jmath} + 6\,\hat{k}$.

A simplified equation would be nice ☺

ANSWER

B) The evaluation of the 2-form $2\,dx \wedge dy - 3\,dz \wedge dx$ on the ordered pair of vectors:

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$$

ANSWER

C) The angle between the following two vectors in $\mathbb{R}^4$:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

ANSWER

D) The transpose of the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 5 \end{bmatrix}$$

ANSWER

DO NOT

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PROBLEM 12: Consider the following probability density function on the unit ball in $\mathbb{R}^3$ given by $x^2 + y^2 + z^2 \leq 1$.

$$p = \kappa(x^2 + y^2 + z^2)^{\frac{3}{2}}$$

A) What value of $\kappa > 0$ makes p a probability density on this [unit ball] domain? (HINT: there's a reason why I used p for the density instead of the usual "ρ"...)

ANSWER

B) Set up, ***but do not solve***, an integral for the probability that a randomly chosen point in this domain (chosen with this probability density) has $z \geq 0$. Be sure to put in the limits of integration correctly!

C) Use your head and tell me – without computing the integral in part (B) – what is the probability and why?

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PROBLEM 13: Consider the following path integral

$$\int \left(x^2y + \frac{1}{3}y^3\right)\, dx + x\, dy$$

integrated over a circle of radius R, centered at the origin, counterclockwise oriented.

ANSWER

A) Compute the integral over this circle of radius R. Think first!

B) Using your answer to part A) above, what value of $R > 0$ maximizes the integral? Think! And show me that you still remember basic single-variable calculus…

ANSWER

DO NOT WRITE HERE

PROBLEM 14: Consider the following functions:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - x^2 \\ xy \\ y - z \end{pmatrix} \qquad \& \qquad g\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} uv \\ w - u \\ v^2 w \end{pmatrix}$$

A) Compute $[Df]$ and $[Dg]$:

$[Df] =$ $\qquad\qquad$ $[Dg] =$

B) If $u = 1, v = 2, w = 3$, what are the outputs $g(u, v, w)$?
Write your answer as a vector.

ANSWER

C) If $u = 1, v = 2, w = 3$, and these inputs are changing at rates $\dot{u} = 2, \dot{v} = 0,$ & $\dot{w} = -1$, at what rates are the outputs of $f \circ g$ changing? Write your answer as a vector.

ANSWER

Some space for you to work/think/draw…